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The Signum function method for the generation of correlated dichotomic chains

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Abstract
We analyze the signum-generation method for creating random dichotomic sequences with prescribed correlation properties. The method is based on a binary mapping of the convolution of continuous random numbers with some function originated from the Fourier transform of a binary correlator. The goal of our study is to reveal conditions under which one can construct binary sequences with a given pair correlator. Our results can be used in the construction of superlattices and waveguides with selective transport properties.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The study of properties of disordered complex systems with spatial and/or temporal correlations is one of the hot topics in modern physics. Recently, much attention was paid to the related problem of how to construct disordered materials with specific transport properties that are due to underlying correlations in a disorder. One of the important applications of this problem is a creation of electron nano-devices, optic fibers, rough surfaces, acoustic and electromagnetic waveguides with selective transport properties.

It is known (see, for instance, [1–4]) that many properties of systems with weak disorder are determined by the binary (pair or two-point) correlation function of a corresponding random process. Recently, it was found [5–7] that specific long-range correlations in disordered potentials can lead to anomalous transport properties. To date, there exist a number

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of algorithms for generating long-range correlated sequences with prescribed correlations. Among such algorithms the most widespread one is the *convolution method* [6–16]. In this method random elements in the generated chain can be of any value from $-\infty$ to ∞. However, in many applications it is more convenient to construct the sequences with a finite number of random values. An important example of such a system, being occurred in nature, is a sequence of nucleotides in a DNA molecule, consisting of four elements only.

The simplest case of a random sequence of finite elements is a stochastic *dichotomic* (binary) chain of only two elements. In contrast to the case of sequences of continuous elements, the problem of a construction of binary sequences with given correlation properties turns out to be tricky. As was recently shown in [16], there is a serious restriction on the type of pair correlators in binary sequences, in contrast to the sequences with continuous distribution of their elements. Many related results for binary sequences can be found in [17–19].

A direct way to create dichotomic correlated sequences is to apply the signum function to the sequence of continuous values, obtained with the convolution method. This method is based on the convolution of a white-noise sequence with some function that is determined by the desired pair correlator. However, as was numerically found in [20], the created binary sequence turns out to have the pair correlator different from the expected one. To date, it remains unclear how to construct binary sequences having the same pair correlators as in the sequences with continuous values of their elements. In spite of its quite simple form, the signum function method is not rigorously analyzed in the literature.

The present paper makes an effort to clarify this problem. Our aim is to understand under what conditions the discussed method allows us to construct binary sequences with a desired pair correlator. We perform a detailed theoretical analysis of the restrictions arising for binary sequences with long-range correlations. Main attention is paid to the stepwise power spectrum resulting in a power decay of correlations. This type of correlation is extremely important in various applications, such as a creation of devices with selective transport properties.

The paper is organized as follows. Section 2 is devoted to general properties of binary sequences. In particular, we derive some of the conditions restricting the form of pair correlators that a binary sequence can have. In section 3, we describe in detail the signum-generation method and derive basic relations needed for a further analysis. In section 4 we analyze the case of a balanced (unbiased) dichotomic sequence, i.e. the sequence with the zero mean value. Here we also display the restrictions related to the discussed method. Our main findings are reported in section 5 where we consider the most interesting case of long-range correlations with the stepwise spectrum. In section 6 we ask a question about general restrictions appearing in the case of power decay of correlations for binary sequences. In the last section we give some additional remarks concerning binary sequences, and summarize our results. The appendices contain some of the details of the analytical and numerical calculations.

2. Necessary conditions

Let us start with generic properties of dichotomic sequences, not associated with a specific choice of a generation method. In view of constructing the sequences with a given pair correlator, it is of great importance to know what are restrictions on the type of correlators allowed for dichotomic sequences. To shed light on this problem, here we consider a statistically homogeneous random sequence of symbols s_n consisting of the values '-1' and '1',

$$s_n = \{-1, 1\}, \qquad n \in \mathbf{Z} = \ldots, -2, -1, 0, 1, 2, \ldots \tag{1}$$

The canonical definition of the correlation function reads as follows,

$$C_s(r) \equiv \overline{s_n s_{n+r}} - \overline{s}^2 = C_s(0) K_s(r), \tag{2}$$

where $\overline{s} \equiv \overline{s_n}$ and $C_s(0) \equiv \overline{s_n^2} - \overline{s}^2$ are the mean value and variance of s_n, respectively. Taking into account a peculiar property

$$\overline{s_n^2} = s_n^2 = 1 \tag{3}$$

of our dichotomic sequence s_n, one obtains

$$C_s(0) = 1 - \overline{s}^2. \tag{4}$$

It should be emphasized that the direct relation between the mean value and higher one-point moments is a specific property of *any* dichotomic chain. For instance, for the binary chain ε_n consisting of '0' and '1' we have the relation $C_\varepsilon(0) = \overline{\varepsilon}(1 - \overline{\varepsilon})$ since $\varepsilon_n^2 = \varepsilon_n$ in this case. In contrast, for a sequence of continuous random numbers of the Gaussian type the variance and mean value are independent parameters.

To proceed with the two-point moments, we associate the correlator $C_s(r)$ with the probabilities of two symbols with the same or opposite signs, occurring at the distance r,

$$4P(\pm 1, \underbrace{\ldots}_{r-1}, \pm 1) = (\overline{s} \pm 1)^2 + C_s(r), \tag{5a}$$

$$4P(\pm 1, \underbrace{\ldots}_{r-1}, \mp 1) = 1 - \overline{s}^2 - C_s(r). \tag{5b}$$

These relationships are also peculiar properties solely of dichotomic sequences, they are drastically distinct from the analogous relations for other random processes (see, e.g. equation (B.8) for the two-point probability density of the Gaussian process). This fact is strictly confirmed by straightforward calculations of equations (5).

In order to obtain expression (5*a*), one should write the average $\overline{(s_n \pm 1)(s_{n+r} \pm 1)}$ via correlator (2),

$$\begin{aligned}\overline{(s_n \pm 1)(s_{n+r} \pm 1)} &= \overline{s_n s_{n+r}} \pm \overline{s_n} \pm \overline{s_{n+r}} + 1 \\ &= C_s(r) + (\overline{s} \pm 1)^2.\end{aligned} \tag{6}$$

On the other hand, the same average can be calculated with the use of two-symbol probabilities,

$$\begin{aligned}\overline{(s_n + 1)(s_{n+r} + 1)} &= 2 \cdot 2 \cdot P(1, \underbrace{\ldots}_{r-1}, 1) + 0 \cdot 2 \cdot P(-1, \underbrace{\ldots}_{r-1}, 1) + 2 \cdot 0 \cdot P(1, \underbrace{\ldots}_{r-1}, -1) \\ &\quad + 0 \cdot 0 \cdot P(-1, \underbrace{\ldots}_{r-1}, -1) = 4P(1, \underbrace{\ldots}_{r-1}, 1).\end{aligned} \tag{7}$$

$$\begin{aligned}\overline{(s_n - 1)(s_{n+r} - 1)} &= 0 \cdot 0 \cdot P(1, \underbrace{\ldots}_{r-1}, 1) \\ &\quad + (-2) \cdot 0 \cdot P(-1, \underbrace{\ldots}_{r-1}, 1) + 0 \cdot (-2) \cdot P(1, \underbrace{\ldots}_{r-1}, -1) \\ &\quad + (-2) \cdot (-2) \cdot P(-1, \underbrace{\ldots}_{r-1}, -1) = 4P(-1, \underbrace{\ldots}_{r-1}, -1).\end{aligned} \tag{8}$$

Then, the combination of equations (6)–(8) results in equality (5*a*).

Similarly, in order to derive expression (5*b*), one can write,

$$\begin{aligned}\overline{(s_n \pm 1)(s_{n+r} \mp 1)} &= \overline{s_n s_{n+r}} \mp \overline{s_n} \pm \overline{s_{n+r}} - 1 \\ &= C_s(r) + \overline{s}^2 - 1.\end{aligned} \tag{9}$$

Again, the above average can be calculated employing two-symbol probabilities. Specifically,

$$\begin{aligned}\overline{(s_n + 1)(s_{n+r} - 1)} &= 2 \cdot 0 \cdot P(1, \underbrace{\ldots}_{r-1}, 1) + 0 \cdot 0 \cdot P(-1, \underbrace{\ldots}_{r-1}, 1) + 2 \cdot (-2) \cdot P(1, \underbrace{\ldots}_{r-1}, -1) \\ &\quad + 0 \cdot (-2) \cdot P(-1, \underbrace{\ldots}_{r-1}, -1) = -4P(1, \underbrace{\ldots}_{r-1}, -1).\end{aligned} \tag{10}$$

$$\begin{aligned}\overline{(s_n - 1)(s_{n+r} + 1)} &= 0 \cdot 2 \cdot P(1, \underbrace{\ldots}_{r-1}, 1) + (-2) \cdot 2 \cdot P(-1, \underbrace{\ldots}_{r-1}, 1) + 0 \cdot 0 \cdot P(1, \underbrace{\ldots}_{r-1}, -1) \\ &\quad + (-2) \cdot 0 \cdot P(-1, \underbrace{\ldots}_{r-1}, -1) = -4P(-1, \underbrace{\ldots}_{r-1}, 1).\end{aligned} \tag{11}$$

From equations (9)–(11) equality (5*b*) follows.

Now, with the use of expressions (5*a*) we express the correlation function $C_s(r)$ via the probabilities to occur three symbols,

$$C_s(r) + (\overline{s} \pm 1)^2 = 4 \sum_{a=-1,1} P(\pm 1, \underbrace{\ldots}_{r'-1}, a, \underbrace{\ldots}_{r-r'-1}, \pm 1). \tag{12}$$

Probability $P(\pm 1, \underbrace{\ldots}_{r'-1}, a, \underbrace{\ldots}_{r-r'-1}, \pm 1)$ is smaller than both probabilities $P(\pm 1, \underbrace{\ldots}_{r'-1}, a)$ and $P(a, \underbrace{\ldots}_{r-r'-1}, \pm 1)$. Thus, we can write

$$\begin{aligned}C_s(r) + (\overline{s} \pm 1)^2 &\leqslant 4 \sum_{a=-1,1} \min\{P(\pm 1, \underbrace{\ldots}_{r'-1}, a), P(a, \underbrace{\ldots}_{r-r'-1}, \pm 1)\} \\ &= \min\{1 - \overline{s}^2 - C_s(r'), 1 - \overline{s}^2 - C_s(r - r')\} \\ &\quad + \min\{(\overline{s} \pm 1)^2 + C_s(r'), (\overline{s} \pm 1)^2 + C_s(r - r')\} \\ &= 1 - \overline{s}^2 + \min\{-C_s(r'), -C_s(r - r')\} \\ &\quad + (\overline{s} \pm 1)^2 + \min\{C_s(r'), C_s(r - r')\}.\end{aligned} \tag{13}$$

Here we again have used equations (5). Then, according to the evident relation

$$\min\{x, y\} + \min\{-x, -y\} = -|x - y|, \tag{14}$$

we arrive at the condition

$$|C_s(r') - C_s(r - r')| + C_s(r) \leqslant 1 - \overline{s}^2. \tag{15}$$

Finally, it is convenient to rewrite equation (15) for the normalized correlator $K_s(r)$,

$$|K_s(r') - K_s(r - r')| + K_s(r) \leqslant 1. \tag{16}$$

Although we have derived this inequality for $0 < r' < r$, a simple analysis reveals its validity for arbitrary values of r and r'. We would like to stress that condition (16) is applicable for any mean value $\overline{s}$ of the dichotomic sequence s_n. However, without a loss of generality in what follows we consider binary sequences with the zero mean, since the statistical properties of considered sequences do not depend on mean values.

In a similar manner one can obtain a second inequality with the use of equation (5*b*),

$$\begin{aligned}1-\overline{s}^2-C_s(r) &= 4\sum_{a=-1,1} P(\pm 1, \underbrace{\ldots}_{r'-1}, a, \underbrace{\ldots}_{r-r'-1}, \mp 1)\\ &\leqslant 4\sum_{a=-1,1} \min\{P(\pm 1\underbrace{\ldots}_{r'-1}, a), P(a, \underbrace{\ldots}_{r-r'-1}, \mp 1)\}\\ &= \min\{1-\overline{s}^2-C_s(r'), (\overline{s}\mp 1)^2+C_s(r-r')\}\\ &\quad + \min\{(\overline{s}\pm 1)^2+C_s(r'), 1-\overline{s}^2-C_s(r-r')\}. \end{aligned} \tag{17}$$

Since $C_s(r) = K_s(r)$ in the case of $\overline{s} = 0$, equation (17) gets a simpler form,

$$|K_s(r') + K_s(r-r')| - K_s(r) \leqslant 1 \qquad \text{for} \quad \overline{s} = 0. \tag{18}$$

Thus, applying the necessary conditions (16) and (17) or (18) if $\overline{s}$=0, one can identify the functions that *cannot be* treated as binary correlators of a dichotomic sequence.

Inequalities (16)–(18) have to be met for any values of indices r and r' and, therefore, they actually represent an infinite set of necessary conditions. Evidently, in every particular case one should choose the strongest condition. On the other hand, equations (16)–(18) are automatically satisfied if one of the indices equals zero, $r = 0$ or $r' = 0$. The same takes place when $r = r'$. Summarizing all these facts, we can combine equations (16) and (18) as follows:

$$\begin{gathered}\max\{|K_s(r') \pm K_s(r-r')| \mp K_s(r)\} \leqslant 1,\\ r \neq 0, \qquad r' \neq 0, \qquad r \neq r' \qquad \text{for} \qquad \overline{s} = 0.\end{gathered} \tag{19}$$

The symbol $\max\{\ldots\}$ implies the absolute maximum of a function with respect to the indices r, r'. Note that equation (19) is automatically fulfilled for two limit cases, namely, for the delta-correlated (white noise) chain, and for the sequences with infinitely long-range correlations. As one can see, for both these cases $K_s(r) = \delta_{r,0}$ or $K_s(r) = 1$, respectively.

3. Binary versus Gaussian

Now we analyze the construction of a dichotomic sequence γ_n by means of the *signum function generation* (SFG) method. It uses an intermediate correlated disorder β_n obtained as a convolution of the uncorrelated Gaussian noise α_n and the modulation function $G(n)$. Specifically, the γ-sequence is defined by

$$\gamma_n = \mathrm{sign}(\beta_n), \tag{20a}$$

$$\beta_n = \overline{\beta} + \sum_{n'=-\infty}^{\infty} G(n-n')\alpha_{n'}. \tag{20b}$$

The initial Gaussian white-noise chain consists of stochastic variables α_n with the zero mean, unit variance and corresponding probability density. Respectively,

$$\overline{\alpha} = 0, \qquad \overline{\alpha_n\alpha_{n'}} = \delta_{n,n'}, \tag{21a}$$

$$\rho_A(\alpha_n = \alpha) = \frac{1}{\sqrt{2\pi}}\exp(-\alpha^2/2). \tag{21b}$$

The bar over a random symbol or function implies the stochastic average.

Evidently, the constructed dichotomic sequence γ_n does not change if one normalizes the intermediate sequence β_n by an arbitrary factor. Hence, without any loss of generality, we can admit the variance of β_n be equal to unity. Then, $\overline{\beta}$ is the mean value of β_n and its correlation function $K_\beta(r)$,

$$K_\beta(r) = \overline{(\beta_n - \overline{\beta})(\beta_{n+r} - \overline{\beta})}, \tag{22}$$

is normalized to unity, $K_\beta(0) = 1$. By a direct substitution of equation (20*b*) into definition (22), the correlator $K_\beta(r)$ is readily associated with the modulation function $G(n)$,

$$K_\beta(r) = \sum_{n=-\infty}^{\infty} G(r-n)G(n). \tag{23}$$

Owing to evenness of $K_\beta(r) = K_\beta(-r)$ and in accordance with equation (23), one can also restrict the function $G(n)$ to the class of even functions, $G(-n) = G(n)$. Note that the condition $K_\beta(0) = 1$ gives rise to the following normalization for $G(n)$,

$$K_\beta(0) = \sum_{n=-\infty}^{\infty} G^2(n) = 1. \tag{24}$$

It is convenient to pass to the Fourier transform of equation (23) with the use of the standard expressions,

$$K_\beta(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}k\, \mathcal{K}_\beta(k)\exp(\mathrm{i}kr), \tag{25a}$$

$$\mathcal{K}_\beta(k) = \sum_{r=-\infty}^{\infty} K_\beta(r)\exp(-\mathrm{i}kr). \tag{25b}$$

The Fourier transform $\mathcal{K}_\beta(k)$ of the pair correlator $K_\beta(r)$ introduced here, is known as the *power spectrum* of random β-chain. Since the correlator $K_\beta(r)$ is real and even function of r, the power spectrum (25*b*) is real, even, $\mathcal{K}_\beta(-k) = \mathcal{K}_\beta(k)$, and non-negative function of the wave number k. Analogously to equation (25), for the modulation function $G(n)$ one can define its Fourier transform $\mathcal{G}(k)$, which is also real and even function, $\mathcal{G}(-k) = \mathcal{G}(k)$.

The Fourier representation of equation (23) reads

$$\mathcal{K}_\beta(k) = \mathcal{G}^2(k). \tag{26}$$

Thus, we arrive at the following expression for the modulation function $G(n)$,

$$G(n) = \frac{1}{\pi}\int_0^{\pi} \mathrm{d}k\, \mathcal{K}_\beta^{1/2}(k)\cos(kn). \tag{27}$$

Evidently, solution (27) automatically satisfies the normalization condition (24).

Since the initial chain α_n is a delta-correlated Gaussian noise, the intermediate variables β_n also constitute a Gaussian random sequence with single-point distribution function $\rho_B(\beta)$ (see appendix A),

$$\rho_B(\beta_n = \beta) = \frac{1}{\sqrt{2\pi}}\exp[-(\beta - \overline{\beta})^2/2]. \tag{28}$$

In order to reveal statistical properties of the signum-generated dichotomic sequence γ_n, one should associate its mean value $\overline{\gamma}$, variance $C_\gamma(0)$ and pair correlator $C_\gamma(r)$ with the corresponding independent characteristics, namely, the mean value $\overline{\beta}$ and the modulation function $G(n)$ (or, the same, with the intermediate correlator $K_\beta(r)$). According to the definition of average, one can express the mean value $\overline{\gamma}$ in terms of $\overline{\beta}$ via the error function [21],

$$\overline{\gamma} \equiv \overline{\gamma_n} = \int_{-\infty}^{\infty} \mathrm{d}\beta\, \rho_B(\beta)\mathrm{sign}(\beta)$$
$$= \sqrt{\frac{2}{\pi}} \int_0^{\overline{\beta}} \mathrm{d}x\, \exp(-x^2/2) \equiv \mathrm{erf}(\overline{\beta}/\sqrt{2}). \tag{29}$$

Similar to equation (4), the variance $C_\gamma(0)$ is written in the form

$$C_\gamma(0) \equiv \overline{\gamma_n^2} - \overline{\gamma}^2 = 1 - \overline{\gamma}^2. \tag{30}$$

An important characteristic of the stochastic sequence γ_n is the correlation function $C_\gamma(r)$,

$$C_\gamma(r) \equiv \overline{\gamma_n\gamma_{n+r}} - \overline{\gamma}^2 = C_\gamma(0)K_\gamma(r). \tag{31}$$

Its calculation is performed in appendix B. Here we write down only the final equation that relates the correlator $K_\gamma(r)$ to $K_\beta(r)$,

$$(1 - \overline{\gamma}^2)K_\gamma(r) = \frac{2}{\pi} \int_0^{K_\beta(r)} \frac{\mathrm{d}x}{\sqrt{1 - x^2}} \exp\left(-\frac{\overline{\beta}^2}{1 + x}\right). \tag{32}$$

Note that the rhs of the latter equation is not an elementary function, therefore its analytical study is not simple. However, the case $\overline{\beta} = 0$ allows one to perform complete analytical analysis.

4. Unbiased sequence

If the mean value of the intermediate chain β_n vanishes, $\overline{\beta} = 0$, then due to equation (29), the mean value of the generated sequence γ_n also vanishes, $\overline{\gamma} = 0$. In this case relation (32) turns out to be remarkably simplified,

$$K_\gamma(r) = \frac{2}{\pi} \int_0^{K_\beta(r)} \frac{\mathrm{d}x}{\sqrt{1 - x^2}} = \frac{2}{\pi} \arcsin[K_\beta(r)]. \tag{33}$$

Another equivalent form is

$$K_\beta(r) = \sin\left[\frac{\pi}{2} K_\gamma(r)\right]. \tag{34}$$

From the above relations one can conclude that the γ-sequence generated with the discussed signum function method, is random. Indeed, the decay of correlations with an increase of $|r|$ in the intermediate β-chain, also results in the decay of correlations in the generated dichotomic γ-sequence.

Now it is suitable to rewrite equation (34) in the Fourier representation,

$$\mathcal{K}_\beta(k) = \mathcal{S}\{K_\gamma\}(k). \tag{35}$$

Here the symbol $\mathcal{S}\{\cdot\}(k)$ stands for the operator that transforms the function $K(r)$ by the following rule,

$$\mathcal{S}\{K\}(k) \equiv \sum_{r=-\infty}^{\infty} \sin\left[\frac{\pi}{2}K(r)\right] \exp(-\mathrm{i}kr) \tag{36a}$$

$$= \left(1 - \frac{\pi}{2}\right) + \frac{\pi}{2}\mathcal{K}(k) + 2\sum_{r=1}^{\infty} \left\{ \sin\left[\frac{\pi}{2}K(r)\right] - \frac{\pi}{2}K(r)\right\} \cos(kr). \tag{36b}$$

It is important to note that series (36*a*) can converge very slowly. Therefore, in the analytical and numerical analysis one has to take into account a lot of terms in the sum in order to obtain correct result. To avoid this problem, we have used the following trick that is based on the second equality (36*b*). Namely, since $K(r) \to 0$ when $|r| \to \infty$, the latter sum converges quite rapidly according to the asymptotic relation

$$\sin\left[\frac{\pi}{2}K(r)\right] - \frac{\pi}{2}K(r) \to \frac{\pi^3}{48}K^3(r), \qquad |r| \to \infty. \tag{37}$$

The substitution of equation (35) into equation (27) yields the following final relation between the modulation function $G(n)$ and the correlator $K_\gamma(r)$ of the generating dichotomic noise γ_n,

$$G(n) = \frac{1}{\pi}\int_0^\pi \mathrm{d}k \sqrt{\mathcal{S}\{K_\gamma\}(k)}\cos(kn). \tag{38}$$

Since the correlator $K_\gamma(r)$ is supposed to be known, relation (38) should be regarded as the expression determining the modulation function $G(n)$.

As one can see, the SFG method for constructing the correlated dichotomic sequence γ_n with the zero mean, unit variance and the prescribed two-point correlator $K_\gamma(r)$ reduces to the following steps. First, starting from a desirable profile of $K_\gamma(r)$ and employing equations (36) and (38), one has to obtain the modulation function $G(n)$. After, the correlated sequence γ_n can be generated in accordance with equation (20). However, it is important to take into account the restriction that directly follows from equation (35). Specifically, since the power spectrum of any random process, in particular $\mathcal{K}_\beta(k)$, is a non-negative function of the wave number k, the function $\mathcal{S}\{K_\gamma\}(k)$ also has to be non-negative,

$$\mathcal{S}\{K_\gamma\}(k) \geqslant 0 \qquad \text{for} \quad |k| \leqslant \pi. \tag{39}$$

This condition becomes apparent from equation (38), in which the function $\mathcal{S}\{K_\gamma\}(k)$ enters as a radicand. In other words, with this method the function $K_\gamma(r)$ can be considered as a correlator of a dichotomic random sequence γ_n, if and only if $\mathcal{S}\{K_\gamma\}(k)$ is a non-negative function of k.

In view of the revealed restriction (39), let us make a qualitative analysis of equation (36*b*). Note that the first summand $(1 - \pi/2)$ is negative. Therefore, if the spectrum $\mathcal{K}(k)$ vanishes within some interval of k and the whole function $\mathcal{S}\{K\}(k)$ is non-negative there, consequently, the third summand is positive, exceeding the value $(\pi/2 - 1)$. However, in the third summand only first few terms determine its sign and give correct estimate of the magnitude. Thus, one may expect that the correlation function of a dichotomic chain, whose spectrum is zero in some interval, cannot be generated by the SFG method.

It should be stressed that condition (39) is a quite serious restriction to the type of correlators allowed for dichotomic sequences. As a demonstration, let us consider a simple example of the one-step additive Markov chain of variables $\varepsilon_n = \{0, 1\}$ obtained according to the conditional probability

$$P(\varepsilon_n = 1|\varepsilon_{n-1}) = \overline{\varepsilon} + (\varepsilon_{n-1} - \overline{\varepsilon})\exp(-k_c), \tag{40}$$

where the parameter k_c is the inverse correlation length. As is known [22], this chain has an exponential pair correlator, $K_{\exp}(r) = \exp(-k_c|r|)$. It is instructive that the function $\mathcal{S}\{K_{\exp}\}(k)$ takes negative values for small k_c below $k_c^* \approx 1.099\ldots$, see the data in figure 1. Thus, the SFG method can reproduce exponential correlator only for $k_c \geqslant k_c^*$, however, not for small k_c in the most interesting region of long-range correlations.

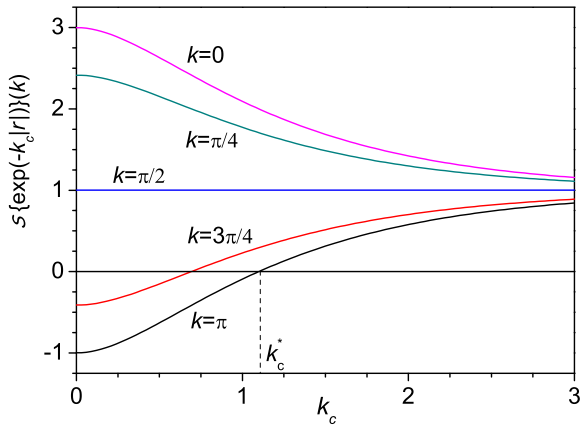

Figure 1. Dependence $\mathcal{S}\{\exp(-k_c|r|)\}(k)$ on the correlation parameter k_c for several values of wave number k.

5. Long-range correlators with stepwise spectrum

5.1. Maximal jump

Here we demonstrate that the discussed method cannot be applied for a construction of dichotomic sequences with long-range correlators resulting in the stepwise power spectrum

$$K_\gamma(r) = \frac{\sin(k_c r)}{k_c r}, \tag{41a}$$

$$\mathcal{K}_\gamma(k) = \frac{\pi}{k_c}\Theta(k_c - |k|), \qquad 0 < k_c \leqslant \pi, \qquad |k| \leqslant \pi. \tag{41b}$$

This kind of correlations is of specific interest in view of applications to 1D disordered superlattices with a selective transport, see, e.g. [6, 7]. Here, k_c is the correlation parameter (inverse correlation length) to be specified, and $\Theta(x)$ implies the Heaviside unit-step function, $\Theta(x < 0) = 0$ and $\Theta(x > 0) = 1$.

First, we analyze the values of k_c within the interval $0 < k_c < \pi$. Two specific cases of $k_c = 0$ and $k_c = \pi$ will be considered afterwards.

In the analysis of the SFG method the crucial characteristic is the function $\mathcal{S}\{K_\gamma\}(k)$ defined by equation (36). It has to be non-negative for any value of the argument k within the interval $|k| \leqslant \pi$, see equation (39). In the case of the long-range correlator (41*a*) it is suitable to use the following explicit expression for $\mathcal{S}\{K_\gamma\}(k)$,

$$\mathcal{S}\left\{\frac{\sin(k_c r)}{k_c r}\right\}(k) = \left(1 - \frac{\pi}{2}\right) + \frac{\pi^2}{2k_c}\Theta(k_c - |k|)$$
$$+2\sum_{r=1}^{\infty}\left\{\sin\left[\frac{\pi}{2}\frac{\sin(k_c r)}{k_c r}\right] - \frac{\pi}{2}\frac{\sin(k_c r)}{k_c r}\right\}\cos(kr). \tag{42}$$

It is remarkable that the summand in the last term behaves as $\pi^3/24k_c^3 r^3$ when $r \to \infty$. Hence, at finite k_c the sum converges quite rapidly and *uniformly*. Therefore, the sum is a *continuous* function of k, in particular, at $k = k_c$, and can be easily calculated numerically.

In figure 2 the behavior of the radicand (42) in equation (38) is shown for several values of the inverse correlation length k_c. Since $\mathcal{S}\{\cdot\}(k)$ is an even function of the wave number k, the discussion can be restricted by the interval $0 \leqslant k \leqslant \pi$. From equation (42) and figure 2

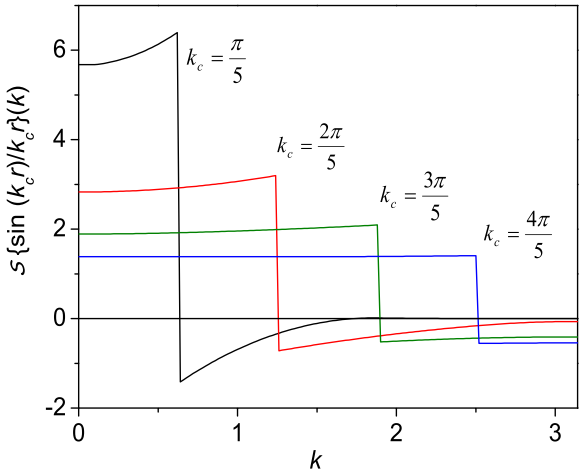

Figure 2. Dependence $\mathcal{S}\{\sin(k_c r)/k_c r\}(k)$ on k for different values of k_c.

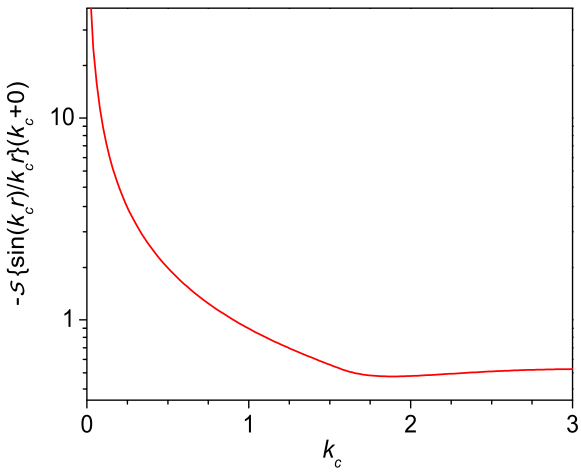

Figure 3. Dependence $-\mathcal{S}\{\sin(k_c r)/k_c r\}(k_c + 0)$ versus k_c in log-scale.

one can draw the following conclusions.

(1) Due to the last term in expression (42) the function $\mathcal{S}\{\sin(k_c r)/k_c r\}(k)$ increases with an increase of k for all k_c within both intervals $(0, k_c)$ and (k_c, π).

(2) The negative jump of $\mathcal{S}\{\sin(k_c r)/k_c r\}(k)$ occurs at $k = k_c$, at the same point where the power spectrum (41*b*) has a jump. The maximal and minimal values of the function are achieved at $k = k_c - 0$ and $k = k_c + 0$, respectively. The jump is exclusively formed by the second term in equation (42). Therefore, its value reads

$$\mathcal{S}\{\sin(k_c r)/k_c r\}(k = k_c - 0) - \mathcal{S}\{\sin(k_c r)/k_c r\}(k = k_c + 0) = \pi^2/2k_c. \tag{43}$$

(3) The minimal value $\mathcal{S}\{\sin(k_c r)/k_c r\}(k = k_c + 0)$ is negative for all finite values of k_c within the interval $0 < k_c < \pi$. The complete dependence of the positive function $-\mathcal{S}\{\sin(k_c r)/k_c r\}(k_c + 0)$ on k_c is depicted in figure 3.

Item (3) displays clearly that for all finite values $k_c < \pi$ the correlator and corresponding power spectrum (41) cannot be created by making use of the discussed method. Indeed, the modulation function $G(n)$ turns out to be of complex value, see equation (38).

Now we determine the values of k_c for which function (41*a*) cannot be the pair correlator of a dichotomic sequence regardless of the method of generation. This can be done with the use of the results of section 2.

First, we demonstrate analytically that this function cannot be the correlation function of a dichotomic sequence s_n with arbitrary mean value $\overline{s}$ for small but finite values of k_c, namely, for $0 < k_c \ll 1$. To this end, we take equation (16) at $r' = 1$ and $r = r_a = [a/k_c]$, where $[x]$ is the integer part of x and a is a constant,

$$\left|\frac{\sin k_c}{k_c} - \frac{\sin k_c(r_a - 1)}{k_c(r_a - 1)}\right| + \frac{\sin k_c r_a}{k_c r_a} \leqslant 1. \tag{44}$$

Being expanded in small parameter k_c, this condition reads

$$\left(\frac{\cos a}{a} - \frac{\sin a}{a^2}\right) k_c + O\left(k_c^2\right) \leqslant 0. \tag{45}$$

It is evident that the lhs of equation (45) can be positive at some values of a (e.g. for $a = 2\pi + \pi/4$). Thus, requirement (16) is violated.

The inequality (18) can also result in new restrictions for allowed values of k_c for the case $\overline{s} = 0$. Rewriting it at $r = 2$ and $r' = 1$, we get

$$4\sin k_c - \sin 2k_c \leqslant 2k_c. \tag{46}$$

Numerical analysis shows that this condition does not hold true for all finite k_c from the interval $0 < k_c < k^*$, where $k^* \approx 2.139\ldots$. It is met only at $k_c > k^*$. However, since equation (18) is just a necessary condition, one cannot guarantee an existence of correlator (41*a*) even at $k_c > k^*$.

At the critical point $k_c = \pi$ the correlator and spectrum (41) reduce to $K_\gamma(r) = \delta_{r,0}$ and $\mathcal{K}_\gamma(k) = 1$. This gives rise to the relations, $\mathcal{S}\{\delta_{r,0}\}(k) = 1$ and $G(n) = \delta_{n,0}$, hence, $\beta_n = \alpha_n$. Consequently, one can apply the SFG. However, this specific case of $k_c = \pi$ is not interesting because from the Gaussian white-noise α_n-sequence the method fabricates the dichotomic chain γ_n again of white-noise type.

As to the singular point $k_c = 0$, here we have $K_\gamma(r) = 1$ for the correlator, and $\mathcal{K}_\gamma(k) = 2\pi\delta(k)$ for the power spectrum, therefore, the radicand is $\mathcal{S}\{1\}(k) = 2\pi\delta(k)$. One can see that the SFG is formally applicable. Besides, the necessary conditions (45) and (46) are satisfied automatically. However, this case is a singular one since for any arbitrarily small but finite values of k_c it is not possible to create a dichotomic sequence with the correlation properties (41) neither by the SFG or by any other method.

5.2. Partial jump

Now we extend our analysis to a more general correlation function that may have various applications. This function also results in a stepwise power spectrum, however, with an additional parameter h that determines the height of step,

$$K_{\gamma,h}(r) = h\delta_{r,0} + (1-h)\frac{\sin(k_c r)}{k_c r}, \tag{47a}$$

$$\begin{aligned} &\mathcal{K}_{\gamma,h}(k) = h + (1-h)\frac{\pi}{k_c}\Theta(k_c - |k|) > 0, \\ &0 \leqslant h \leqslant 1, \qquad 0 < k_c \leqslant \pi, \qquad |k| \leqslant \pi. \end{aligned} \tag{47b}$$

Equation (47) coincides with equation (41) if the step-parameter $h = 0$. Otherwise, when $h = 1$ the generated γ-sequence turns into a dichotomic white noise independently of k_c. Also, γ_n becomes delta correlated at $k_c = \pi$ for arbitrary h. Therefore, at $k_c = \pi$ the conclusions of the previous subsection are also valid.

In what follows, it is convenient to analyze finite values of $k_c < \pi$. The power spectrum (47*b*) is an even function of the wave number k and has two symmetric jumps at the points $k = \pm k_c$. For positive $0 < k \leqslant \pi$ the spectrum abruptly falls down at $k = k_c$ from the maximal value $\mathcal{K}_{\gamma,h}(k < k_c) = h + (1-h)\pi/k_c$ to the minimal one, $\mathcal{K}_{\gamma,h}(k > k_c) = h$. Evidently, this jump can be regarded as a mobility edge of disordered 1D conductors, if the maximal value $h + (1-h)\pi/k_c$ is much larger than the minimal one, h,

$$1 + \frac{1-h}{h}\frac{\pi}{k_c} \gg 1. \tag{48}$$

One can see that for finite h well above zero, this is possible only for small k_c. What is more tricky, for $k_c \ll 1$ the mobility edge may emerge even in the case when the generated γ-sequence is close to a dichotomic white noise, i.e. when $1 - h \ll 1$. Therefore, one should have,

$$0 < k_c \ll 1 - h \ll 1. \tag{49}$$

The reason of existence of the mobility edge under conditions (49) is a significant contribution of the second term in coorelator (47*a*). In spite of the fact that it has a quite small amplitude $1 - h \ll 1$, this term provides extremely long-range correlations with the characteristic scale $k_c^{-1} \gg (1-h)^{-1} \gg 1$.

Now we address the function $\mathcal{S}\{K_{\gamma,h}\}(k)$ that must be non-negative in order to construct the correlated sequence γ_n with the use of the SFG method. In accordance with definition (36), an appropriate analysis can be done with the following explicit expression,

$$\begin{aligned}\mathcal{S}\{K_{\gamma,h}\}(k) = 1 - \frac{\pi}{2}(1-h) + (1-h)\frac{\pi^2}{2k_c}\Theta(k_c - |k|)\\ +2\sum_{r=1}^{\infty}\left\{\sin\left[\frac{\pi}{2}(1-h)\frac{\sin(k_c r)}{k_c r}\right] - \frac{\pi}{2}(1-h)\frac{\sin(k_c r)}{k_c r}\right\}\cos(kr).\end{aligned} \tag{50}$$

As in the previous case (41), the summand in the last term behaves as $\pi^3/24k_c^3 r^3$ if $r \to \infty$. Hence, at finite k_c the sum converges rapidly and *uniformly*. It is a *continuous* function of k, in particular, at $k = k_c$.

The numerical calculations of $\mathcal{S}\{K_{\gamma,h}(r)\}(k)$ performed for finite $0 < k_c < \pi$ and $0 \leqslant k \leqslant \pi$, are shown in figure 4. Together with equation (50) they provide us with the following empirical results.

(1) Due to the last term in equation (50), the function $\mathcal{S}\{K_{\gamma,h}\}(k)$ increases with k at arbitrary values of k_c and h within both intervals $(0, k_c)$ and (k_c, π).

(2) The negative jump of $\mathcal{S}\{K_{\gamma,h}\}(k)$ occurs at the same point $k = k_c$ as for the jump of the prescribed power spectrum (47*b*). The maximal and minimal values of the function are reached, respectively, at $k = k_c - 0$ and $k = k_c + 0$ for all values of the step-parameter h within $0 \leqslant h < 1$. The jump is exclusively related to the third term in equation (50). Therefore, its value reads

$$\mathcal{S}\{K_{\gamma,h}\}(k_c - 0) - \mathcal{S}\{K_{\gamma,h}\}(k_c + 0) = (1-h)\frac{\pi^2}{2k_c}. \tag{51}$$

(3) Depending on h, the minimum $\mathcal{S}\{K_{\gamma,h}\}(k_c + 0)$ can be either negative or positive. Also, the value of $\mathcal{S}\{K_{\gamma,h}\}(k_c + 0)$ monotonically increases with an increase of h.

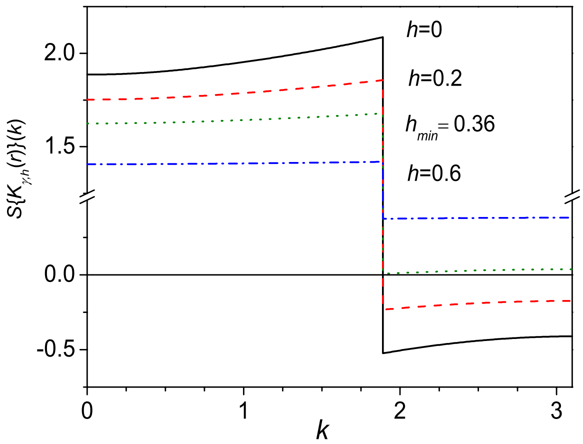

Figure 4. Function $\mathcal{S}\{K_{\gamma,h}(r)\}(k)$ versus k for few values of h at $k_c = 1.89$ (this k_c is very close to the minimum point of the curve in figure 3 and to the minimum point of the upper curve in figure 5). The function is entirely positive if $h > h_{\min}$. When $h = h_{\min}$, the function goes to zero solely at one point $k = k_c + 0$ and is positive otherwise. For $h < h_{\min}$, the function is negative within the whole interval $k_c < k < \pi$.

The presented numerical analysis can be supplemented with the following two points. First, from the treatment of the case (41), we know that

$$\mathcal{S}\{K_{\gamma,h}\}(k_c + 0) < 0 \qquad \text{for} \quad h = 0. \tag{52}$$

Second, from definition (50) one can easily reveal that

$$\mathcal{S}\{K_{\gamma,h}\}(k) = 1 \qquad \text{for} \quad h = 1. \tag{53}$$

Summarizing our results, one can conclude that there exists a threshold value of the step-parameter h that we refer to as $h_{\min}$, that separates the region $0 \leqslant h < h_{\min}$ in which $\mathcal{S}\{K_{\gamma,h}\}(k)$ has negative values, from the region with non-negative values,

$$\mathcal{S}\{K_{\gamma,h}\}(k) \geqslant 0 \qquad \text{for} \quad h_{\min} \leqslant h \leqslant 1. \tag{54}$$

It is clear that the threshold $h_{\min}$ obeys the equation (see figure 4)

$$\mathcal{S}\{K_{\gamma,h_{\min}}\}(k_c + 0) = 0, \tag{55}$$

and depends on the correlation parameter k_c. By substitution of equation (50) into equation (55), one can rewrite it in the explicit form,

$$\frac{\pi}{4}(1 - h_{\min}) - \frac{1}{2} = \sum_{r=1}^{\infty} \left\{ \sin\left[\frac{\pi}{2}(1 - h_{\min})\frac{\sin(k_c r)}{k_c r}\right] - \frac{\pi}{2}(1 - h_{\min})\frac{\sin(k_c r)}{k_c r} \right\} \cos(k_c r). \tag{56}$$

The numerical solution $h_{\min}(k_c)$ of this equation is displayed in figure 5 by the upper curve.

Equation (56) can be solved analytically at small k_c, resulting in

$$h_{\min}(k_c) = 1 - \left(\frac{96 k_c}{\pi^4}\right)^{1/3} + O\left(k_c^{2/3}\right) \qquad \text{for} \quad k_c \ll 1. \tag{57}$$

This expression exhibits the limit $h_{\min} \to 1$ for $k_c \to 0$.

Thus, in accordance with the study performed above, the dichotomic sequence γ_n with the long-range correlator and the stepwise power spectrum (47) can be constructed by the SFG

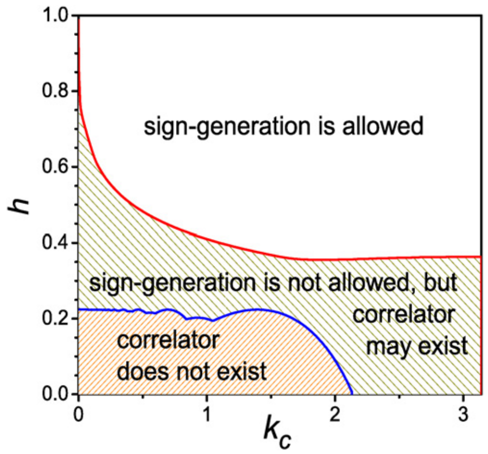

Figure 5. Space of control parameters k_c and h. The upper curve is the dependence $h_{\min}(k_c)$ while lower curve depicts $h_0(k_c)$. Within the lowest area in which $0 \leqslant h < h_0$, a dichotomic sequence γ_n with the correlator $K_{\gamma,h}(r)$ does not exist. The area with $h_{\min} \leqslant h \leqslant 1$ allows for a creation of γ_n with $K_{\gamma,h}(r)$ by the discussed method. In the intermediate region $h_0 \leqslant h < h_{\min}$ the SFG method does not work, and an existence of a dichotomic sequence with the stepwise spectrum remains an open problem.

method only if its parameters k_c and h are placed onto or above the upper line in figure 5. Only in this area of the (k_c, h)-plane does condition (54) hold true. Unfortunately, practically within this whole area the parameters k_c and h have values of the order of one, and, therefore, one cannot satisfy a quite strong requirement (48) in order to clearly observe a mobility edge. The only exception is a narrow vicinity of the point $(k_c = 0, h = 1)$. Remarkably, in this vicinity due to specific dependence (57) of $h_{\min}(k_c)$, conditions (49) can be satisfied,

$$0 < k_c \ll 1 - h \leqslant 1 - h_{\min}(k_c) \approx \left(\frac{96k_c}{\pi^4}\right)^{1/3} \ll 1, \tag{58}$$

and, consequently, the mobility edge can be achieved. So, equation (58) gives us the only (perhaps, just purely theoretical) possibility of arranging a mobility edge in the transport through the γ-sequence constructed by the SFG method.

Now we analyze the consequences of the necessary conditions formulated in section 2. For the long-range correlator (47*a*) inequalities (19) lead to the following restriction with respect to the step-parameter h,

$$1 - \max{}^{-1}\{R(r, r')\} \leqslant h \tag{59}$$

at $r \neq 0$, $r' \neq 0$ and $r \neq r'$. Here we have introduced the function

$$R(r, r') = \left|\frac{\sin k_c r'}{k_c r'} \pm \frac{\sin k_c (r - r')}{k_c (r - r')}\right| \mp \frac{\sin k_c r}{k_c r}. \tag{60}$$

Since by the definition $h \geqslant 0$, requirement (59) is meaningful only if its lhs is positive. Otherwise, it is satisfied automatically. The combination of equation (59) with the assumption $0 \leqslant h \leqslant 1$ gives rise to the relation,

$$h_0 \leqslant h \leqslant 1, \tag{61}$$

where new characteristic quantity h_0 is introduced,

$$h_0 = 1 - \max{}^{-1}\{1, \max\{R(r, r')\}\}. \tag{62}$$

This function $h_0(k_c)$ is shown in figure 5 by the lower curve. Its piecewise shape is caused by the fact that different r and r' contribute to h_0 within different intervals of k_c. Finally, when k_c becomes equal or larger than $k^* \simeq 2.139\ldots$ (see the text after equation (46)), we have $h_0 = 0$ and the necessary conditions (61) reduce to the initial ones, $0 \leqslant h \leqslant 1$.

Thus, taking into account that the required area of the parameters k_c and h is determined by the relation, $0 < k_c < \pi$, $0 \leqslant h \leqslant 1$ of the (k_c, h)-plane, one can summarize the following. A dichotomic sequence with a long-range correlator and a stepwise power spectrum (47) does not exist within the lowest region $0 \leqslant h < h_0(k_c)$. When $h_0(k_c) \leqslant h < h_{\min}(k_c)$, the dichotomic chain cannot be created by the SFG method and there is no answer whether it can be created by any other method. Finally, within the highest zone with $h_{\min}(k_c) \leqslant h \leqslant 1$, one can construct desired dichotomic sequences with the use of the discussed method.

5.3. Predefined intermediate spectrum

With the SFG method, we first generate intermediate Gaussian sequence β_n and after, the dichotomic sequence γ_n. Above, we specified the correlator $K_\gamma(r)$ of a final dichotomic γ-sequence and analyzed the spectrum $\mathcal{K}_\beta(k) = \mathcal{S}\{K_\gamma\}(k)$ of the intermediate Gaussian β-chain, keeping in mind that the latter must be non-negative, see equations (35), (38).

Below, we ask question about the type of the spectrum of γ_n, that emerges if the intermediate Gaussian sequence β_n is assumed to have given pair correlator with stepwise power spectrum of the following form,

$$K_\beta(r) = \frac{\sin(k_c r)}{k_c r}, \tag{63a}$$

$$\mathcal{K}_\beta(k) = \frac{\pi}{k_c}\Theta(k_c - |k|), \qquad 0 < k_c \leqslant \pi, \qquad |k| \leqslant \pi. \tag{63b}$$

In accordance with relation (33) and Fourier transforms (25), the corresponding correlator and power spectrum of the dichotomic γ-sequence read,

$$K_\gamma(r) = \frac{2}{\pi}\arcsin\left[\frac{\sin(k_c r)}{k_c r}\right], \tag{64a}$$

$$\mathcal{K}_\gamma(k) = \frac{2}{\pi}\sum_{r=-\infty}^{\infty}\arcsin\left[\frac{\sin(k_c r)}{k_c r}\right]\exp(-\mathrm{i}kr)$$
$$= \left(1 - \frac{2}{\pi}\right) + \frac{2}{k_c}\Theta(k_c - |k|) + \frac{4}{\pi}\sum_{r=1}^{\infty}\left\{\arcsin\left[\frac{\sin(k_c r)}{k_c r}\right] - \frac{\sin(k_c r)}{k_c r}\right\}\cos(kr). \tag{64b}$$

Here, last representation for the spectrum $\mathcal{K}_\gamma(k)$ is similar to that we have employed for the function $\mathcal{S}\{\cdot\}(k)$ in its analysis (compare with equations (36) and (42)). It is noteworthy to emphasize that the summand in the last term of this representation behaves as $2\big/3\pi k_c^3 r^3$ when $r \to \infty$. Hence, as above, the sum converges rapidly and *uniformly*. Therefore, it can be easily calculated numerically.

Figure 6 presents the behavior of the power spectrum (64*b*) for dichotomic γ_n in the case of the predefined stepwise profile (63*b*) for the spectrum of intermediate sequence β_n. Since $\mathcal{K}_\gamma(k)$ is an even function of the wave number k, the presentation is sufficient within $0 \leqslant k \leqslant \pi$. Assuming $k_c < \pi$, from equation (64*b*) and figure 6 one can conclude:

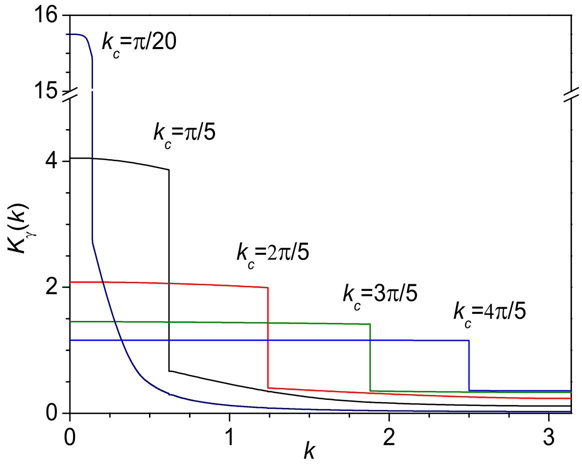

Figure 6. Power spectrum $\mathcal{K}_\gamma(k)$ versus k for different values of the correlation parameter k_c.

(1) Function (64*b*) is positive within the whole interval $|k| \leqslant \pi$. Therefore, it truly serves as a power spectrum and its inverse Fourier transform (64*a*) is a valid correlator of the generated sequence γ_n;

(2) Due to the second and last terms in equation (64*b*) the spectrum $\mathcal{K}_\gamma(k)$ decreases with an increase of k for all k_c within the whole interval $(0, \pi)$;

(3) The negative jump of $\mathcal{K}_\gamma(k)$ occurs at $k = k_c$, at the same point as for the jump of the prescribed intermediate power spectrum $\mathcal{K}_\beta(k)$. The maximal and minimal values of $\mathcal{K}_\gamma(k)$ at the jump are defined at $k = k_c - 0$ and $k = k_c + 0$, respectively. The jump is exclusively formed by the second term in equation (64*b*). Therefore, its value reads

$$\mathcal{K}_\gamma(k_c - 0) - \mathcal{K}_\gamma(k_c + 0) = 2/k_c; \tag{65}$$

(4) The value $\mathcal{K}_\gamma(k = k_c - 0)$ to the left from the jump is always positive. The value $\mathcal{K}_\gamma(k = k_c + 0)$ to its right is also positive for all k_c. Their ratio $\mathcal{K}_\gamma(k_c - 0)/\mathcal{K}_\gamma(k_c + 0)$ being of the order of one for finite k_c, seems to slightly increase with a decrease of k_c. However, it is saturated if the inverse correlation length k_c vanishes. Indeed, from equation (64*b*) one can easily get,

$$\frac{\mathcal{K}_\gamma(k_c - 0)}{\mathcal{K}_\gamma(k_c + 0)} = \frac{\pi + I(k_c) + (\pi/2 - 1)k_c}{I(k_c) + (\pi/2 - 1)k_c} \tag{66a}$$

$$\to \frac{\pi + I(0)}{I(0)} = 6.0096\ldots \qquad \text{if} \quad k_c \to 0. \tag{66b}$$

Here we have introduced

$$I(k_c) = 2k_c \sum_{r=1}^{\infty} \left\{ \arcsin\left[\frac{\sin(k_c r)}{k_c r}\right] - \frac{\sin(k_c r)}{k_c r} \right\} \cos(k_c r); \tag{67a}$$

$$I(0) = 2\int_0^\infty \mathrm{d}x \left\{ \arcsin\left[\frac{\sin x}{x}\right] - \frac{\sin x}{x} \right\} \cos x. \tag{67b}$$

In spite of the divergence of the jump (65), the convergence of the ratio (66) at $k_c \to 0$ occurs because the values of the spectrum $\mathcal{K}_\gamma(k)$ to the left, $\mathcal{K}_\gamma(k_c - 0)$, and to the right,

$\mathcal{K}_\gamma(k_c+0)$, from the jump, increase with decreasing of k_c exactly in the same manner, $\mathcal{K}_\gamma(k_c-0)\approx 2[\pi+I(0)]/\pi k_c$ and $\mathcal{K}_\gamma(k_c+0)\approx 2I(0)/\pi k_c$.

In addition, it can be analytically shown that in the limit $k_c\to 0$, the correlator $K_\gamma(r)$ tends to unity, while the spectrum $\mathcal{K}_\gamma(k)$ turns into the Dirac delta-function,

$$\lim_{k_c\to 0} K_\gamma(r) = \frac{2}{\pi}\lim_{k_c\to 0}\arcsin\left[\frac{\sin(k_c r)}{k_c r}\right] = 1; \tag{68a}$$

$$\lim_{k_c\to 0}\mathcal{K}_\gamma(k) = 2\pi\delta(k). \tag{68b}$$

Therefore, as the correlation parameter k_c vanishes, the final dichotomic γ-sequence has extremely long-range correlations.

On the contrary, for $k_c=\pi$ the correlator $K_\gamma(r)$ reduces to the Kronecker delta-symbol, whereas the power spectrum $\mathcal{K}_\gamma(k)$ degenerates into unity,

$$K_\gamma(r) = \frac{2}{\pi}\arcsin[\delta_{r,0}] = \delta_{r,0}; \tag{69a}$$

$$\mathcal{K}_\gamma(k) = 1 \qquad \text{for} \quad k_c=\pi. \tag{69b}$$

Thus, the final dichotomic γ_n reduces to the white-noise chain.

In summary, in spite of the fact that the jump ratio $\mathcal{K}_\gamma(k_c-0)/\mathcal{K}_\gamma(k_c+0)$ in the power spectrum $\mathcal{K}_\gamma(k)$ is of the order of unity for all $k_c<\pi$, we hope that the mobility edge can be observed at small enough correlation parameter $k_c\ll 1$, due to specific delta-function behavior of $\mathcal{K}_\gamma(k)$ itself (see figure 6 and equation (68*b*)).

6. Power correlators

6.1. Power correlator for dichotomic sequence

Here we consider an important problem of constructing a sequence with the power correlation function and corresponding spectrum,

$$K_{\gamma,p}(r) = \delta_{r,0} + (k_c|r|)^{-p}(1-\delta_{r,0}), \qquad \mathcal{K}_{\gamma,p}(k) = 1 \tag{70a}$$

$$+\,k_c^{-p}\{\mathrm{Li}_p[\exp(\mathrm{i}k)] + \mathrm{Li}_p[\exp(-\mathrm{i}k)]\}, \qquad p>0, \quad k_c\geqslant 1, \quad |k|\leqslant\pi. \tag{70b}$$

Here p and k_c are positive real numbers characterizing how fast the correlator decreases. Note that the parameter k_c cannot be less than one since $K_{\gamma,p}(r)\leqslant 1$. The Fourier transform $\mathcal{K}_{\gamma,p}(k)$ of this correlator is expressed via the polylogarithm function $\mathrm{Li}_q(z)$ that is defined by

$$\mathrm{Li}_q(z) = \sum_{r=1}^{\infty}\frac{z^r}{r^q}. \tag{71}$$

For $K_{\gamma,p}(r)$ to be the correlator of a stochastic process, it is necessary to have $\mathcal{K}_{\gamma,p}(k)\geqslant 0$ for all k. This condition is satisfied if and only if the following inequality is fulfilled,

$$k_c \geqslant [-2\mathrm{Li}_p(-1)]^{1/p}. \tag{72}$$

This result is due to the shape of spectrum $\mathcal{K}_{\gamma,p}(k)$ that monotonously decreases with an increase of k within the interval $(0,\pi)$, and reaches its minimal value at $k=\pi$. The rhs of condition (72) can be calculated in limit cases,

$$[-2\mathrm{Li}_p(-1)]^{1/p} = \begin{cases}\pi/2 - cp, & p\ll 1,\\ 1+p^{-1}\ln 2, & p\gg 1,\end{cases} \qquad c = 1.029\ldots. \tag{73}$$

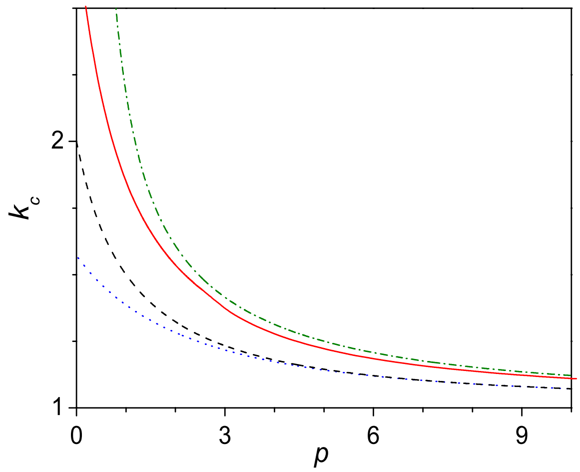

Figure 7. Various borders of the parameters p and k_c above which the following relations are fulfilled: (*a*) equation (72) (dotted curve), (*b*) equation (74) (solid curve), (*c*) equation (80) (dashed curve), (*d*) equation (84) (dash-dotted curve).

In figure 7 the area where the power spectrum (70*b*) is non-negative, is located above the dotted lowest curve.

If $K_{\gamma,p}(r)$ is the correlator of a *dichotomic* random sequence generated by the SFG method, then $\mathcal{S}\{K_{\gamma,p}\}(k) \geqslant 0$ for all values of k. Since this function, as well as $\mathcal{K}_{\gamma,p}(k)$, monotonously decreases, the only condition required,

$$\mathcal{S}\{K_{\gamma,p}\}(\pi) \geqslant 0. \tag{74}$$

Expanding in equation (36*a*) the sin-function into series, we get useful expression

$$\mathcal{S}\{K_{\gamma,p}\}(k) = 1 + \sum_{l=0}^{\infty} \frac{\pi^{2l+1}}{2^{2l+1}(2l+1)!} k_c^{-p(2l+1)} \{\mathrm{Li}_{p(2l+1)}[\exp(\mathrm{i}k)] + \mathrm{Li}_{p(2l+1)}[\exp(-\mathrm{i}k)]\}. \tag{75}$$

For $p \to \infty$ it can be approximately calculated as

$$\mathcal{S}\{K_{\gamma,p}\}(k) \simeq 1 + 2\sin\left(\pi k_c^{-p}/2\right)\cos k. \tag{76}$$

Therefore, taking into account that $k_c \geqslant 1$, one obtains the following asymptotic of condition (74)

$$k_c \geqslant 1 + p^{-1}\ln 3 \qquad \text{for} \quad p \gg 1. \tag{77}$$

The area where the discussed SFG method is applicable located in figure 7 above solid curve.

Now we analyze the necessary condition for the existence of the power correlator (70*a*) for the dichotomic sequence regardless of the generation method. For the sake of simplicity we consider the case $\overline{\gamma} = 0$. From inequalities (19) one can obtain the following relation,

$$\max\{||r'|^{-p} \pm |r-r'|^{-p}| \mp |r|^{-p}\} \leqslant k_c^p, \qquad r \neq 0, \quad r' \neq 0, \quad r \neq r'. \tag{78}$$

It is easy to see that the maximum with respect to r', occurs at $r' = \pm 1$ or $r' = r \pm 1$. Therefore, equation (78) can be rewritten as

$$\max_{r>0}\{1 \pm (r+1)^{-p} \mp r^{-p}\} \leqslant k_c^p. \tag{79}$$

The last condition is equivalent to

$$k_c \geqslant (2-2^{-p})^{1/p} = \begin{cases} 2-2p\ln^2 2, & p \ll 1, \\ 1+p^{-1}\ln 2, & p \gg 1. \end{cases} \tag{80}$$

In figure 7 the border corresponding to this condition is designated by dashed curve. It should be noted that equation (80) is the necessary condition, thus, if it is met, it is still not clear whether the correlator with such values of parameters p and k_c exists.

6.2. *Predefined intermediate power correlator*

We have found that the dichotomic sequence γ_n with the power correlator (70*a*) can be constructed by the SFG method for some values of parameters p and k_c that meet the condition (74). Now let us take the intermediate Gaussian sequence β_n prescribed to have the power correlator and the corresponding spectrum

$$K_{\beta,p}(r) = \delta_{r,0} + (k_c|r|)^{-p}(1-\delta_{r,0}), \tag{81a}$$

$$\mathcal{K}_{\beta,p}(k) = 1 + k_c^{-p}\{\mathrm{Li}_p[\exp(\mathrm{i}k)] + \mathrm{Li}_p[\exp(-\mathrm{i}k)]\},$$
$$p > 0, \quad k_c \geqslant 1, \quad |k| \leqslant \pi. \tag{81b}$$

Evidently, condition (72) is implied to be met. In accordance with relation (33) and Fourier transforms (25), the correlator and power spectrum of the generated dichotomic γ-sequence are described as

$$K_{\gamma,p}(r) = \delta_{r,0} + \frac{2}{\pi}\arcsin[(k_c|r|)^{-p}](1-\delta_{r,0}), \tag{82a}$$

$$\mathcal{K}_{\gamma,p}(k) = 1 + \frac{4}{\pi}\sum_{r=1}^{\infty}\arcsin[(k_c r)^{-p}]\cos(kr). \tag{82b}$$

We can assert that since equation (72) is satisfied, i.e. the spectrum (81*b*) of the intermediate β-sequence is non-negative, then the spectrum (82*b*) of the generated dichotomic γ_n is also non-negative.

When $|r| \to \infty$, the correlator (82*a*) tends to zero in accordance with the following asymptotic:

$$K_{\gamma,p}(r) \backsimeq (k_c'|r|)^{-p}, \qquad k_c' = k_c(\pi/2)^{1/p}. \tag{83}$$

As was shown, the allowed values of k_c are expressed by equation (72). Therefore, the scaling parameter k_c' should satisfy the condition

$$k_c' \geqslant [-\pi\mathrm{Li}_p(-1)]^{1/p}. \tag{84}$$

This condition for possible values of k_c' and p is met in the area above the dash-dotted curve in figure 7.

Thus, the mapping of the Gaussian sequence with the power correlation function (81*a*) into the binary sequence result in the same power for the decrease of the final correlator (82*a*) expressed by equation (83). However, such behavior of the final correlator occurs only asymptotically, for sufficiently large values of $|r|$.

7. Conclusion

First, we would like to emphasize the following point that was briefly mentioned in the beginning. Our study of the correlation properties of a random dichotomic sequence γ_n is based on example (1) in which the two elements are '−1' and '1'. On the other hand, there is a simple correspondence between this chain and a dichotomic sequence $\varepsilon(n)$ consisting of two arbitrary symbols ε_0 and ε_1,

$$\varepsilon(n) = \{\varepsilon_0, \varepsilon_1\}, \qquad n \in \mathbf{Z} = \ldots, -2, -1, 0, 1, 2, \ldots \tag{85}$$

The correspondence is expressed by the linear relationship,

$$\varepsilon(n) = \frac{\varepsilon_0 + \varepsilon_1}{2} \mp \frac{\varepsilon_0 - \varepsilon_1}{2}\gamma_n. \tag{86}$$

The choice of the sign is not important. It determines only into what symbol, ε_0 or ε_1, the initial values '−1' and '1' are converted.

In accordance with equation (86) and due to specific properties (30), (3) of the γ-sequence, the connection between the mean values and variances is as follows,

$$\varepsilon^2(n) = \frac{\varepsilon_0^2 + \varepsilon_1^2}{2} \mp \frac{\varepsilon_0^2 - \varepsilon_1^2}{2}\gamma_n; \tag{87a}$$

$$\overline{\varepsilon} = \frac{\varepsilon_0 + \varepsilon_1}{2} \mp \frac{\varepsilon_0 - \varepsilon_1}{2}\overline{\gamma}; \tag{87b}$$

$$C_\varepsilon(0) \equiv \overline{\varepsilon^2(n)} - \overline{\varepsilon}^2 = \frac{(\varepsilon_0 - \varepsilon_1)^2}{4}C_\gamma(0). \tag{87c}$$

Analogously, the two-point correlation function $C_\varepsilon(r)$ of the ε-chain is associated with the binary correlation function $C_\gamma(r)$ of the sequence γ_n as follows:

$$C_\varepsilon(r) \equiv \overline{\varepsilon(n)\varepsilon(n+r)} - \overline{\varepsilon}^2 = \frac{(\varepsilon_0 - \varepsilon_1)^2}{4}C_\gamma(r). \tag{88}$$

The comparison of equations (87*c*) and (88) makes obvious the equality between the normalized correlators $K_\varepsilon(r)$ and $K_\gamma(r)$,

$$K_\varepsilon(r) \equiv C_\varepsilon(r)/C_\varepsilon(0) = C_\gamma(r)/C_\gamma(0) \equiv K_\gamma(r). \tag{89}$$

Thus, our analysis is valid for any dichotomic sequence.

Our results can be summarized as follows. We have shown that the statistical properties of random dichotomic sequences are principally different from those known for sequences with a continuous distribution of their elements. We were able to find analytically conditions (19) that can be used to know whether a binary sequence can have the desired pair correlator. Note that only these conditions are necessary.

Another important restriction is due to inequality (39) derived under quite general assumptions. We have shown that even in the well-known case of an exponential decay of correlations, there are no binary sequences that can be created with the SFG method, unless the decay is sufficiently strong. This fact is very important in view of many applications.

Our specific interest was in a possibility of creating, with the considered method, the binary sequences with long-range correlations described by equations (41*a*) and (41*b*). We have analytically found that function (41*a*) cannot be a pair correlator of any binary sequence. We have also examined a more general correlation function (see equation (47)) that corresponds to the generalization of the stepwise power spectrum. Our extensive examination of the signum function method, applied to this correlation function, has revealed the regions of parameters

k_c and h for which the pair correlator can emerge in a binary sequence. Correspondingly, we identified the regions where such a pair correlator cannot appear in binary sequences. For other values of the control parameters we cannot give definite answer, therefore, a further study is needed.

Finally, we analyzed an important case of the power decay of the pair correlator. Recently, the problem of the generation of random processes with power correlations has attracted much attention in the literature. Analyzing such correlators, we have found that the SFG method in principle allows us to construct binary sequences with these correlators, however, with some restrictions on the values of parameters in equation (70).

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Appendix A. Probability density of β_n

The standard way to derive the probability density $\rho_B(\beta)$ of the random process β_n is due to its *characteristic function* $\varphi_B(t)$ defined by

$$\varphi_B(t) \equiv \overline{\exp[\mathrm{i}t\beta_n]} = \int_{-\infty}^{\infty} \mathrm{d}\beta\, \rho_B(\beta) \exp(\mathrm{i}t\beta). \tag{A.1}$$

From the last equality in this definition it immediately follows that the probability density $\rho_B(\beta)$ is the Fourier transform of $\varphi_B(t)$,

$$\rho_B(\beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, \varphi_B(t) \exp(-\mathrm{i}t\beta). \tag{A.2}$$

To start with, we substitute the explicit expression (20*b*) for β_n into definition (A.1) for the characteristic function $\varphi_B(t)$. Then, we rewrite the result as an infinite product of exponential functions and take into account the statistical independence of uncorrelated random variables α_n. This procedure yields

$$\overline{\exp[\mathrm{i}t\beta_n]} = \exp(\mathrm{i}t\bar{\beta}) \prod_{n'=-\infty}^{\infty} \overline{\exp[\mathrm{i}tG(n-n')\alpha_{n'}]}. \tag{A.3}$$

In accordance with the Gaussian distribution (21*b*) of α_n, its characteristic function is

$$\varphi_A(\tau) \equiv \overline{\exp(\mathrm{i}\tau\alpha_n)} \equiv \int_{-\infty}^{\infty} \mathrm{d}\alpha\, \rho_A(\alpha) \exp(\mathrm{i}\tau\alpha) = \exp(-\tau^2/2). \tag{A.4}$$

The use of equations (A.3), (A.4) with $\tau = tG(n-n')$, and the normalization condition (24) results in

$$\varphi_B(t) = \exp(\mathrm{i}\bar{\beta}t - t^2/2). \tag{A.5}$$

As is known, this characteristic function corresponds to the Gaussian probability density (28). One can confirm this fact by a direct evaluation of the integral in equation (A.2).

Appendix B. Pair correlator of γ_n

Let us derive the pair correlator $\overline{\gamma_n\gamma_{n+r}}$. Employing the standard integral presentation for the signum function,

$$\mathrm{sign}(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}x \frac{\sin(zx)}{x}, \tag{B.1}$$

and equation (A.4), we arrive, in a manner similar to the calculation of the characteristic function $\varphi_B(t)$ in appendix A, at the expression,

$$\overline{\gamma_n\gamma_{n+r}} = \mathcal{J}(K_\beta(r), \overline{\beta}) = \frac{2}{\pi^2}\int_0^\infty \frac{dx_1}{x_1}\int_0^\infty \frac{dx_2}{x_2}\exp\left(-\frac{x_1^2+x_2^2}{2}\right)\sum_{s=-1,1} s\exp[sK_\beta(r)x_1x_2]\cos[\overline{\beta}(x_1 - sx_2)]. \tag{B.2}$$

Thus, we have reduced the problem to the derivation of $\mathcal{J}(K,\beta)$. To solve it, we obtain the derivative of $\mathcal{J}(K,\beta)$ with respect to K. After some simplifications one gets,

$$\frac{\partial}{\partial K}\mathcal{J}(K,\beta) = \frac{1}{\pi^2}\int_0^\infty dx_1\int_{-\infty}^\infty dx_2 \exp\left(-\frac{x_1^2+x_2^2}{2}\right)\exp[Kx_1x_2]\sum_{t=-1,1}\exp[it\beta(x_1-x_2)].$$

To proceed, we write down the following relation that is valid for arbitrary real quantities a and b,

$$\int_{-\infty}^\infty dx \exp(-x^2/2)\exp(ax)\exp(ibx) = \sqrt{2\pi}\exp[(a^2-b^2)/2]\exp(iab). \tag{B.3}$$

Using equation (B.3) with $a = Kx_1$ and $b = -t\beta$ we integrate over x_2 and make further simplifications,

$$\frac{\partial}{\partial K}\mathcal{J}(K,\beta) = \frac{\sqrt{2\pi}}{\pi^2}\int_{-\infty}^\infty dx_1 \exp[-x_1^2(1-K^2)/2]\exp(-\beta^2/2)\exp[i\beta(1-K)x_1].$$

Now we change the integration variable x_1,

$$x_1' = x_1\sqrt{1-K^2}. \tag{B.4}$$

Then, applying equation (B.3) with

$$a = 0, \qquad b = \beta\sqrt{\frac{1-K}{1+K}}, \tag{B.5}$$

we perform the integration over x_1 that gives rise to the expression

$$\frac{\partial}{\partial K}\mathcal{J}(K,\beta) = \frac{2}{\pi\sqrt{1-K^2}}\exp\left(-\frac{\beta^2}{1+K}\right). \tag{B.6}$$

The general solution of equation (B.6) is

$$\mathcal{J}(K,\beta) = \mathcal{J}(0,\beta) + \frac{2}{\pi}\int_0^K \frac{dx}{\sqrt{1-x^2}}\exp\left(-\frac{\beta^2}{1+x}\right). \tag{B.7}$$

It should be noted that equation (B.7) can be also obtained by means of the two-point probability density that for the correlated Gaussian sequence β_n with the correlator $K_\beta(r)$ is defined by

$$\rho_B(\beta_n=\beta, \beta_{n+r}=\beta') = \frac{1}{2\pi\sqrt{1-K_\beta^2(r)}}\exp\left\{-\frac{\beta^2+\beta'^2-2K_\beta(r)\beta\beta'}{2\left[1-K_\beta^2(r)\right]}\right\}. \tag{B.8}$$

The last step we should take, is to calculate $\mathcal{J}(0,\beta)$. It can be directly obtained from equation (B.2),

$$\mathcal{J}(0,\overline{\beta}) = \left[\frac{2}{\pi}\int_0^\infty dx\frac{\sin(\overline{\beta}x)}{x}\exp(-x^2/2)\right]^2 = \overline{\gamma}^2. \tag{B.9}$$

This result can be easily explained. Indeed, the condition $K_\beta(r) = 0$ implies that the correlations between β_n and β_{n+r} disappear, hence, the correlations between γ_n and γ_{n+r} are absent as well.

As a result of these calculations, we finally get

$$\mathcal{J}(K_\beta(r), \overline{\beta}) = \overline{\gamma}^2 + \frac{2}{\pi} \int_0^{K_\beta(r)} \frac{\mathrm{d}x}{\sqrt{1-x^2}} \exp\left(-\frac{\overline{\beta}^2}{1+x}\right)$$

This expression provides equation (32).

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